NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS.

TECHNICAL MEMORANDUM. 35

4.4
4.7.3

CALCULATION OF WING SPARS.

By

H. Müller-Breslau.

Translated from
"Zeitschrift für Flugtechnik und Motorluftschiffahrt,"
April 30, 1920.

August, 1921.

CALCULATION OF WING SPARS.[1]

By

Müller-Breslau.

In the same issue of this magazine, Mr. Ratzersdorfer[2] refers to my article in the October, 1919, number and leads me to compare and combine the numerical and geometrical determinations of the maximum M.

The object of my 1919 article, which was inspired by Mr. Pröll, was the simplification, not only of the approximate, but also of the exact calculation of the maximum bay moment, whereby especial importance is attached (even with a very low Euler safety factor) to the rapid and accurate calculation of $\Gamma = \pi^2 E J/S s^2$.[3]

The geometric representation of the bay moments consists in multiplications, prescribed by formulas, of lines with circular functions. Mr. Ratzersdorfer starts with the formulas in my "Graphische Statik," which have often been given in this magazine.[4] I will now use formulas developed in my 1919 article, p. 200.

---

1) From "Zeitschrift für Flugtechnik und Motorluftschiffahrt," April 30, 1920, pp. 102-105.
2) See N.A.C.A. file 1116.2-67.
3) The case of lower values of Γ is especially important, since it is customary in airplane construction to calculate with loads near the breaking strength.
   I will soon give a report on the experiments which support this method. They were made, according to a plan worked out by myself, in the Prussian material-testing office, for both bending and breaking, with two bay wooden spars furnished by the former "Flugzeugmeisterei" (Aircraft Inspection and Engineering Department).
4) See also my treatise "Zur Theorie der Biegungsspannungen in Fachwerkträgern (On the Theory of Bending Stresses in Truss Girders), Allgemeine Bauzeitung, 1885.

I.

From the formula

$$M = C_1 \cos \frac{\zeta}{k} + C_2 \sin \frac{\zeta}{k} - g k^2$$

for the moment at the distance ζ from the middle of the bay (Fig. 1), it directly follows that $M + g k^2$ (Fig. 2) can be represented by the projection a d of the right angle a b c, formed by C_1 and C_2, on a straight line which, with C_1, incloses the angle

$$\varphi = \frac{\zeta}{k} = \alpha \frac{\zeta}{s} \qquad \alpha = s \sqrt{\frac{S}{E J}}$$

It follows from the formulas

$$C_2 = \frac{M_A - M_B}{2 \sin \frac{\alpha}{2}} = \frac{1}{2} (M_A - M_B) \operatorname{cosec} \frac{\alpha}{2}$$

$$C_1 = D \sec \frac{\alpha}{2}$$

$$D = g k^2 + \frac{1}{2} (M_A + M_B)$$

that C_2 and C_1, through their projections

$\overline{b c'} = \frac{1}{2} (M_A - M_B)$ and $\overline{a' b} = D$

are determined on a straight line which, with C_1, forms the angle $\frac{\alpha}{2}$ 1).

Almost without exception $M_{max.}$ will now be required. It then suffices to draw the right angle triangle a b c whose hypotenuse $\overline{a c}$ gives the value

$$g k^2 + M_{max.}$$

1) M_A and M_B are positive when, according to Fig. 1, they act in the same direction as g. In my 1919 article, in agreement with Pröll, I represented M_A and M_B as positive relieving moments.

If the absolute value of the angle c a b is greater than $\frac{\alpha}{2}$ then $M_{max.}$ equals the greater of the end moments M_A and M_B.

Formulas (16) and (17) on p.200 (1919) can be combined into the following formula

$$M_{max.} = \sqrt{C_1^2 + C_2^2 - g\,k^2} \quad . . . . (1)$$

which may also be obtained from Fig. 2. Usually $M_{max.}$ lies so near the middle of the bay, that we may write

$$M_{max.} = C_1 + \frac{C_2^2}{2C_1} - g\,k^2 \quad . . . .(2)$$

in agreement with formulas (11), (18) and (20) of my 1919 article. If α lies in the neighborhood of π, then C_1 must be calculated. The formula

$$\sec\frac{\alpha}{2} = \frac{\Gamma + \epsilon}{\Gamma - 1}$$

on p.198 (1919) and the table on p.200, for very small safety factors, lead quickly and accurately to the goal. For a low value of Γ we may write:1)

$$C_2 = \frac{1}{2}(M_A - M_B)$$

---

1) Point d (Fig. 2) lies on a circle, whose diameter is a c. Mr. Ratzersdorfer determines this circle by means of the conditions $M = M_A$ for $x = 0$ and $M = M_B$ for $x = s$. If α varies only a little from π, then the three points determining the circle lie nearly in a straight line. I give the calculation the preference and call the attention of readers, who likewise prefer to calculate, to two tables:

J. Hrabák, "Praktische Hilfstabellen" (Practical Tables) (Published by Teubner in Leipzig). Table V, "Trigonometrische, Linien von Minute zu Minute," also contains the often used secants and cosecants.

C. Burrau, "Tafeln der Functionen Cosinus und sinus mit den natürlichen Zahlen als Argument" (G. Reiner, Berlin).

The secant table with the argument Γ (1919, p.200) also enables the calculation of C_2^2, since

$$\operatorname{cosec}^2 = \frac{\sec^2}{\sec^2 - 1}$$

In the numerical examples of my 1919 article, I made the calculations with excessive accuracy, because I wished to obtain as accurate a comparison as possible of the results obtained by the different formulas. I will therefore give still another example for the accurate determination of $M_{max.}$ together with the round numbers to be recommended in practice.

Let $J = 314$ cm.^{4}, $W = 64$ cm.^{3}, $F = 28$ cm.^{2}, $s = 305$ cm., $S = 3840$ kg., $g = 2$kg/cm., $E = 120{,}000$ kg/cm.^{2}, $M_A = -44.8$ kg.m. $M_B = -324.4$ kg.m., $\frac{1}{2}(M_A + M_B) = -184.6$ kg.m. $\frac{1}{2}(M_A - M_B) = 140$ kg.m.

We obtain:

$$g\,k^2 = g\,\frac{E\,J}{S} = 9812.5\,g = 196.3 \text{ kg m.}$$

$$\Gamma = \frac{\pi^2 k^2}{s^2} = 1.041, \qquad \sec\frac{\alpha}{2} = \frac{1.041 + 0.272}{0.041} = 32.0$$

$$C_1 = (196.3 - 184.6)32.0 = 374 \text{ kg m.} \qquad C_2 = 140 \text{ kg m.}$$

$$M_{max.} = \sqrt{374^2 + 140^2} - 196.3 = 203 \text{ kg m.}$$

$$\sigma = \frac{20300}{64} + \frac{3840}{28} = 454 \text{ kg/cm.}^2$$

For the determination of the flexure, according to the calculation of

$$M_o = g\,\frac{s^2}{8} = 232.6 \text{ kg m.}$$

(Compare 1919, p.198, formulas (7) and (4).) The following formulas serve:

$$y_{max.} < \frac{203 - 232.6 + 184.6}{3840} = 0.040 \text{ m.}$$

$$y_{max.} > \frac{374 - 196.3 - 232.6 + 184.6}{3840} = 0.034 \text{ m.}$$

If the safety factor Γ is very large and α is consequently small, then the moment $g\,k^2 = g\,s^2/\alpha^2$ becomes inconveniently large for geometrical representation. 1)

For example, let

$$\frac{M_A}{M_o} = -0.628,\quad \frac{M_B}{M_o} = -0.096,\quad \frac{M_A + M_B}{2\,M_o} = -0.362,$$

$$\frac{M_A - M_B}{2\,M_o} = -0.266,\quad \Gamma = 25,\quad \frac{\alpha}{2} = \frac{90^\circ}{\sqrt{\sigma}} = 18^\circ,\quad \sec\frac{\alpha}{2} = 1.0515,$$

$$\operatorname{cosec}\frac{\alpha}{2} = 3.236,\quad \frac{g\,k^2}{M_o} = \frac{8}{\alpha^2} = \frac{8\,\Gamma}{\pi} = 20.264.$$

We obtain for $M_o = 1$

$C_1 = (20.264 - 0.362)\ 1.0515 = 20{,}927$, $C_2 = -0.266 \times 3.236 = -0.861$, $M_{max} = 20.927 + \frac{0.861^2}{2 \times 20.927} - 20.264 = 0.681$

---

1) For small values of α, the values used for determining the supporting moments of spars with several bays.

$$\psi' = \frac{v'}{Ss} = \frac{s}{EJ} \times \frac{v'}{\alpha^2},\ \psi'' = \frac{v''}{Ss} = \frac{s}{EJ} \times \frac{v''}{\alpha^2},\ \frac{g\,s\,v'''}{S} = \frac{g\,s^2}{EJ} \times \frac{v'''}{\alpha^2}$$

are calculated with the formulas obtained from a series development

$$\frac{v'}{\alpha^2} = \frac{1}{3} + \frac{\alpha^2}{45},\quad \frac{v''}{\alpha^2} = \frac{1}{6} + \frac{7\alpha^2}{360},\quad \frac{v'''}{\alpha^2} = \frac{1}{24} + \frac{\alpha^2}{240}$$

for whose usefulness the statement vouches, that for $\alpha = 20^\circ$ ($\alpha = 0.34907$) they still give the values 0.33604, 0.16904 and 0.04217, which differ only in the fifth decimal place from the exact values 0.33607, 0.16907 and 0.04218.

In the introduction to the article "Zur Knickungsbiegung," by Konig, in this magazine, 1919, No. 21, I remarked that the case of very small values of α (including $\alpha = 0$) is treated in my "Graphische Statik" Vol. II, Chap. 2, p. 289.

If we disregard the influence of Γ on the moments, we have

$$M_{max.} = 1 - 0.362 + \left(\frac{0.266}{2}\right)^2 = 0.656$$

The calculation gives the result quickly and exactly for all safety factors, but presupposes the use of tables of functions.

II.

In the limit case $\alpha = \pi$, $\sec\frac{\alpha}{2} = \infty$. If the moments to remain finite, D must equal 0. This gives

$$M_A + M_B = -2\ h\ k^2 = -\frac{2\ g\ s^2}{\pi^2}$$

and $C_1 = 0 \times \infty$. At least one of the moments M_A and M_B must be a function of α.[1)]

A general investigation of the value

$$C_1 = \left(D \sec\frac{\alpha}{2}\right)_{\alpha=\pi} = -2\left(\frac{d\,D}{d\,\alpha}\right)_{\alpha=\pi}$$

will be given in a special article. Here I will confine myself to a simple example. I will take a beam (Fig. 3) centrally loaded and resting on three rigid supports, and to whose ends are applied the moments M_A and M_B. If $\sigma_1 > \alpha_2$ then may $\alpha_1 > \pi$. This may happen, without exceeding the proportionality limit.

From the equation

$$M_A \frac{\nu_1''}{S_1 s_1} + M_B\left(\frac{\nu_1'}{S_2 s_2} + \frac{\nu_2''}{S_2 s_2}\right) + M_C \frac{\nu_3''}{S_2 s_2} = -\frac{g_1 s_1 \nu_1'''}{S_1} - \frac{g_2 s_2 \nu_2'''}{S_2}$$

when, for abbreviation, we write

$$\frac{S_1 s_1}{S_2 s_2} = \gamma$$

there follows

$$M_B = \frac{-g_1 s_1^2 \nu_1''' - g_2 s_2^2 \nu_2'''\gamma - M_A \nu_1'' - M_C \nu_2''\gamma}{\nu_1' + \nu_2'\gamma}$$

1) See next page.

The introduction of this value into the expression

$$C_1 = [g_1 k_1^2 + \frac{1}{2}(M_A + M_B)] \sec\frac{\alpha_1}{2}$$

$$= [g_1 \frac{s_1^2}{\alpha_1^2} + \frac{1}{2}(M_A + M_B)] \sec\frac{\alpha_1}{2}$$

gives, after a simple intermediate calculation,

$$2\, C_1 \alpha_1^2 \quad (\sin\alpha_1 - \alpha_1 \cos\alpha_1 + \gamma' \cdot v_2' \sin\alpha_1) =$$

$$= + g_1\, s_1^2 \; [\sin\frac{\alpha_1}{2}(4 + \alpha_1^2 + 2\,\gamma\, v_2') - 2\,\alpha_1 \cos\frac{\alpha_1}{2}]$$

$$- g_2\, s_2^2\, \gamma\, v_2'''\, \alpha_1^2 \sin\frac{\alpha_1}{2}$$

$$+ 2\, M_A\, \alpha_1^2 \; [\sin\frac{\alpha_1}{2}(2 + \gamma\, v_2') - \alpha_1 \cos\frac{\alpha_1}{2}]$$

$$- 2\, M_C\, \gamma\, v_2''\, \alpha_1^2 \sin\frac{\alpha_1}{2}$$

From this follows for $\alpha_1 = \pi$:

$$C_1 = \frac{g_1 s_1^2 \; (4 + \pi^2 + 4\,\gamma\, v_2')}{2\,\pi^3} - \frac{g_2 s_2^2\, \gamma\, v_2'''}{2\,\pi^3}$$

$$+ M_A \frac{2 + \gamma\, v_2'}{\pi} - M_C \frac{\gamma\, v_2''}{\pi} \quad 2)$$

Numerical example.- $g_1 = 2.0$ kg/cm. $s_1 = 330$ cm., $M_A = -25{,}000$ kg/cm., hence, $g_1 s_1 = 217{,}800$ kg/cm. and for $\alpha = \pi$

1) See my articles in this magazine, 1918, Nos. 17 and 18, and in "Zentrabblatt der Bauverwaltung," 1919, No.84. In the latter, the influence of a permanent uneven load is considered.

2) If $s_1 = s_2$, $S_1 = S_2$, and $J_1 = J_2$, then $\alpha_1 = \alpha_2$ may exceed the limit π only for symmetrical loading $g_1 = g_2$ and $M_A = M_C$. We then obtain for $\alpha = \pi$

$$C_1 = \frac{g\, s^2 \; (4 + \pi^2)}{2\,\pi^3} + M_A \; \frac{2}{\pi}$$

$$g_1 \; k_1^2 = \frac{g_1 s_1^2}{\pi^2} = 32{,}068 \text{ kg/cm.}$$

$$M_B = -2 \; g_1 \; k_1^2 - M_A = -19{,}136 \text{ kg/cm.}$$

$$C_2 = \frac{M_A - M_B}{2} = g_1 \; k_1^2 + M_A = -2932 \text{ kg/cm.}$$

The values M_B and C_2 are independent of g_2, s_2, S_2, M_C. The only condition is $\alpha_1 > \alpha_2$ in order that we may have $\alpha_1 > \pi$. If now

$$J_2 = J_1, \quad \frac{s_2}{s_1} = \frac{2}{3}, \quad \frac{S_2}{S_1} = 1.44, \text{ then } \frac{\alpha_2}{\alpha_1} = \frac{2}{3} \quad \sqrt{1.44} = 0.8,$$

$$\gamma = \frac{S_1 s_1}{S_2 s_2} = \frac{25}{24}, \quad \alpha_2 = 0.8 \; \pi = 2.5132741 \; (144^\circ),$$

$$\nu_2{}' = 4.45992250, \quad \nu_2{}'' = 3.2758378, \quad \nu_2{}''' = 0.7245714$$

and we obtain

$$C_1 = 0.52328 \; g_1 s_1^2 - 0.24140 \; g_2 \; s_2^2 + 2.11518 \; M_A - 1.09139 \; M_C$$

$$= 61090 - 11684 \; g_2 - 1.09139 \; M_C.$$

The following table shows the great influence of g_2 and M_C on the maximum bay moment s_1.

g_2 (kg/cm)	M_C (kg/m)	C_1 (kg/m)	$M_{max.}$ Bay s_1 (kg/m)	$M_{max.}$ Bay s_2 (kg/m)
3.0	-80	347.7	128.2	80.8
3.0	-40	304.0	84.8	148.1
3.0	0	260.4	41.4	215.9
2.0	0	377.2	157.6	58.6

We can also determine the true value of $C_1 = 0 \times \infty$ by calculating the value of C_1 for the value of α lying the nearest possible to that of π

We take $\alpha_1 = 180^\circ\ 50'$, $\alpha_2 = 0.8$ $\alpha_1 = 144^\circ\ 40'$ and obtain

$$\nu_1' = -215.9847,\quad \nu_1'' = -218.0077,\quad \nu_1''' = -44.0683$$

$$\nu_2' = +4.5617,\quad \nu_2'' = +3.3659,\quad \nu_2''' = +0.7435$$

$$g_1\ k_1^2 = \frac{g_1 s_1^2}{\alpha_1^2} = 21864.83 \text{ kg/cm.} \qquad \sec\frac{\alpha_1}{2} = -137.511.$$

For the loading case $g_2 = 3$ kg/cm. and $M_C = -80{,}000$ kg/cm., there follows

$$M_B = -19236.95 \text{ kg/cm.,} \qquad D = g_1 k_1^2 + \frac{M_A + M_B}{2} = -253.64 \text{ kg/cm.}$$

$$C_1 = 2.5364 \times 137.511 = 348.8 \text{ kg/m.,} \qquad C_2 = \frac{M_A - M_B}{2} = 28.8 \text{ kg/m.,}$$

$$M_{max.} = C_1 + \frac{C_2^2}{2C_1} - g_1 k_1^2 = 131 \text{ kg/m.}$$

This result varies but slightly from the value 128.2 kg/m. obtained for $\alpha_1 = \pi$. In this comparison, no account is taken of the fact that, with increasing values for α and S, the loads g and the end moments also increase somewhat.

Whither leads, therefore the valuation of the moments in the limit case $\alpha = \pi$, as given by Mr. Ratzersdorfer in his second figure?

The three points a_1, O, b_1, which determine the circle, whose rays o c represent the moments $g\ k^2 + M$, lie, when $\alpha = \pi$, in a straight line. The center of the circle then only remains finite, when the points a_1 and b_1 coincide. This occurs, when $g\ k^2 + M_A = -(g\ k^2 + M_B)$ agrees with the condition $D = o$. Opposite the indefinite form $C_1 = 0 \times \infty$ stands the indefinite task of drawing a circle of which only two points O and

b_1 are given. In his 1919 treatise in the "Osterr. Flugzeitschrift" Mr. Ratzersdörfer takes the distance

$$\overline{o\ b_1} = \frac{1}{2} (M_B - M_A)$$

as the diameter of the circle and obtains the demonstration repeated here in Fig. 4. This gives, for the bay s_1 of our example, moments which are independent of the loading of the bay s_2 and of the moment M_C.

The foregoing investigation presupposes a uniformly loaded bay with a constant end load S. In reality, the air pressure is solved by the ribs into a number of individual loads. Furthermore, the inner tension between the two wing spars results in the application of oblique stresses within the bay of a spar. The bending stresses generated in the spars were, to the best of my knowledge, first considered in print in my treatise on the strength calculation of airplane spars in the August, 1918, number of the Technische Berichte (Technical Bulletin) of the "Flugzeugmeisterei" (Aircraft Inspection and Engineering Department.) I was principally concerned, through the application of general formulas, practicable for every case of loading, in investigating, with a few examples, as to whether the customary assumption, in practice, of a constant pressure and uniform loading, instead of individual loads, is sufficiently accurate, but also in demonstrating, on an important specimen, the great influence of the direction of the stresses resulting from the inner tension. I will go more thoroughly into these questions in a special treatise.

(Translated by the National Advisory Committee for Aeronautics.)

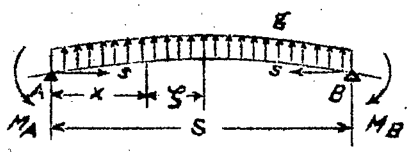

Fig. 1.

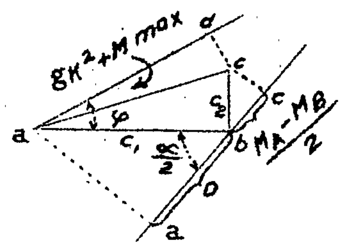

Fig. 2

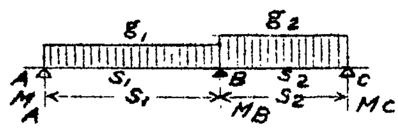

Fig. 3.

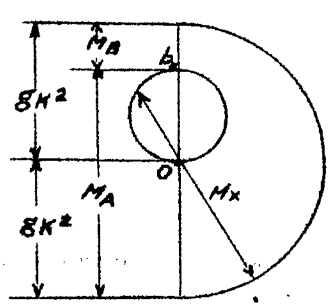

Fig. 4.